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*In blessed memory of A.P. Favorskii*

# Statistical Mechanics of Vortex Hydrodynamic Structures

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**Abstract**—The existing approaches to analyzing the dynamics of coherent vortex structures are considered from the viewpoint of the calculus of variations for Poisson systems. For turbulent hydrodynamic flows with large-scale vortices, a simulation technique in the form of statistical mechanics of the Euler equation in a "coarse-grained" representation is substantiated.

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## 1. INTRODUCTION AND GENERAL REMARKS

The onset and evolution of turbulence in hydrodynamic flows can be described from various positions with the use of widely different mathematical tools. Since this problem is rather complicated, no complete theory for its description has been created until now, although first attempts were made as early as the late 19th century (see [1]). Nevertheless, we can distinguish two basic approaches, which are, in a sense, opposite. They are based on the following antagonistic assumptions: (1) in turbulent flows, we deal with manifestations of stochastic processes (statistical turbulence); and (2) in turbulent flows, there are quasi-deterministic structures on macro- and mesoscales; their dissipation and decay give rise to a collection of unsteady small-scale turbulence processes (methods of statistical turbulence can be locally applied on small scales, although, in principle, there is an alternative, for example, based on the direct solution of the Boltzmann or Katz kinetic equations). The first approach goes back to Keller and Friedmann's works (1925), who gave the general definition of Euler spatiotemporal correlation functions of thermal and hydrodynamic fields in turbulent flows and proposed a method for deriving dynamical equations for correlation functions from the Navier–Stokes equation. These equations are used to average physical flow characteristics, for example, the velocity and the size of the vortices. A mathematical description makes use of the assumption that perturbations on different scales are random and relies on probability theory (generalized to function spaces). This assumption underlies the derivation of simplified equations and is valid when the increments of flow instability development are estimated on different scales. However, observing the velocity variations at a point in turbulent flow, we can conclude that the velocity generally remains within its values in large vortices, since they are the basic elements in the turbulent flow structure. Fast fluctuations occur when small vortices go over large structures. It is this observation that underlies the second approach. Its basic ideas were formulated by Belotserkovskii (on the basis of numerical experiments) in [2, 3] (and were further developed in [4, 5]). According to results of computer simulation, we can believe that large vortices carry the basic energy of turbulent flows and determine the flow structure. The distribution of large vortices does not reflect the random nature of perturbations, but corresponds to the physical laws of fluid dynamics, when the inertial terms in the Navier–Stokes equation prevail over the stresses caused by viscosity (and the equation is reduced to the Euler one). Then the pair of forces arising from the pressure field and dynamical forces related to the velocity field gives rise to a vortex structure. The origin of large ("coherent") vortices and the flow structure have to be described using the theory of the Euler equation.

It seems obvious that the last approach to the description of turbulence (which employs the concept of coherence in a vortex system) has to be justified using the rigorous formalism of the theory of fluid dynamics equations. The fact is that, in terms of the physical interpretation of processes, coherence is related primarily to the preservation of order in an evolving medium in conjunction with the corresponding scale of variations in the spatial characteristics of the system. Actually, this phenomenon regarded as an individual form of organized fluid motion was first described in [6–8]. In modern fluid dynamics, following [9], coherent structures are defined as coupled large-scale turbulent fluid masses with phase-correlated vorticity in the entire spatial region they occupy. Similar definitions are proposed in [10, 11], where the authors believe that the most typical property of coherent structures is that they are isolated from the outer flow in the sense that the vorticity and macroscopic characteristics inside these structures are distributed quasi-deterministically (i.e., they are ordered on average) and, additionally, these structures exist for a rather long time without significant changes in their structure.

The fact that the definition of the physical phenomenon under consideration is weakly formalizable leads to a critical level of uncertainty in its description (the use of numerical coherence criteria as definitions [11, 12] is also hardly justified, since these criteria are not universal and rather arbitrary). Accordingly, to justify the second approach to the simulation of turbulent processes, it is reasonable to invoke the calculus of variations of functionals for Poisson systems [13, 14]. With this mathematical apparatus, the generalized states of relative equilibrium [15, 16] of a hydrodynamic system in the plane can be rigorously defined (as solutions of extremal problems with constraints imposed on a Casimir set) and can be assigned a "coherent structure" understood as a vorticity field solving the Euler–Helmholtz equations. In this paper, for variational problems for the Casimir functionals of a hydrodynamic vortex system, we describe their basic realizations, which have been successfully used and can be of practical interest. More specifically, these realizations are presented in the case of dissipation on small scales or in the case when the system is nonlinearly dynamically stable (the vorticity field produced by variation with a suitable set of additional conditions can be regarded, for example, as an a priori "coherent component" in a triple expansion [17, 18] in the numerical simulation of a multiscale hydrodynamic problem).

## 2. TRANSITION FROM HAMILTONIAN TO POISSON SYSTEMS IN FLUID DYNAMICS

Consider a Hamiltonian system $\{H(\mathbf{x}, \mathbf{p})|\mathcal{M}_N\}$ with a $2N$-dimensional phase space $T^*\mathcal{M}_N = \{(x_k, p_k)\}_{k=1}^{k=N}$. An example is a system of $N$ Onsager vortices [19] in the plane, since its Kirchhoff–Routh function is reduced to Hamiltonian form by renormalization. The fact that the coordinates of the $k$th microstate $(x_k, p_k)$ are canonical implies the validity of the Liouville theorem $\partial\dot{x}_k/\partial x_k + \partial\dot{p}_k/\partial p_k = 0\ \forall k = \overline{1, N}$. With the use of this theorem, we can construct equilibrium statistical mechanics for the given Hamiltonian system, introducing $\mu^N = \prod_{k=\overline{1,N}} dx_k dp_k$ as an invariant measure. For the collection of quantities $\{C_j(\mathbf{x}, \mathbf{p})\}_{j=1}^{j=n}$ conserved in the dynamics of the Hamiltonian system, the quantities $\mu_n^N = (\Omega_n)^{-1}\prod_{k=1}^{N} \Xi(C_1, \dots, C_n) dx_k dp_k$ are also invariant measures (here, $\Xi(\dots)$ is an arbitrary function and $\Omega_n$ is a normalizing constant). For an isolated system, a microcanonical distribution $\mu_{n,\,\mathrm{micro}}^N$ that takes into account the given number of invariants of motion can be introduced following [20] by setting $\Omega_n \longrightarrow \Omega_{n,\,\mathrm{micro}}(C_{1'}, \dots, C_{n'})$ and $\Xi(C_1, \dots, C_n) = \prod_{j=1}^{n} \delta(C_j(\mathbf{x}, \mathbf{p}) - C_{j'})$, in the above general formula. Here, $\Omega_{n,\,\mathrm{micro}}(C_{1'}, \dots, C_{n'})\prod_{j=1}^{n} \Delta C_j$ is the volume of a portion of the phase space such that $C_{j'} \le C_j \le C_{j'} + \Delta C_j\ \forall j$ for sufficiently small $\{\Delta C_j\}$.

The use of the apparatus of microcanonical distributions leads to the sequential construction of equilibrium statistical mechanics for an $N$-particle Hamiltonian system, specifically, for Onsager vortices [21, 22]. To describe the nonstationary evolution of this vortex system, we need to use kinetic theory based on the Liouville equation for the $(2N + 1)$-dimensional distribution function with possible reduction to an equation for a single-vortex distribution function (see, e.g., [23, 24]) by applying the BBGKY method. However, since the kinetic approach is highly complicated, it is reasonable to put it aside for special problems, for example, ones requiring an especially detailed description of the evolution for part of a system, and to invoke the formalism of a canonical assemble for hydrodynamic systems.

In this context, questions arise associated with the fact that the Euler equation is an infinite-dimensional dynamical system and, accordingly, on it we cannot construct a canonical structure (similar to that based on the pair of canonical conjugate variables $(\mathbf{x}, \mathbf{p})$ in the finite-dimensional case). However, following [25, 26], for this system, we can introduce the noncanonical Poisson brackets $\{F_1, F_2\} \equiv \int \omega(\mathbf{x})\,[\delta F_1/\delta\omega, \delta F_2/\delta\omega]d\mathbf{x}$ (here, $F_{1,2}[\omega]$ are functionals, $\delta/\delta\omega$ is the functional derivative, $[G_1(\mathbf{x}), G_2(\mathbf{x})] \equiv (G_1)_{x_1}(G_2)_{x_2} - (G_1)_{x_2}(G_2)_{x_1}$, and $\mathbf{x} = (x_1, x_2)$. With the help of these brackets, the dynamics of the system can be described by an equation of the form $\partial\omega/\partial t = \{\omega, H[\omega]\}$ (where $\omega$ is local vorticity and $H$ is the Hamiltonian of the system). Moreover, there exists an infinite-dimensional set of nuclear (Casimir) functionals $\mathfrak{C}$ of the bracket operator $\{F, \mathfrak{C}\} = 0$ (for an arbitrary functional $F[\omega]$).

The macroscopic state in the considered vortex system in the plane ("reducible to a canonical one") can be defined as follows. From the strictly deterministic vorticity $\omega(\mathbf{x})$ at a given point of the plane, we pass to the probability density $\varrho(\mathbf{x}, \lambda)$ of the existence of a continuous (pseudodiscrete) "vorticity level" $\lambda \in \mathbb{R}^1$ at the point $\mathbf{x} \in D \subseteq \mathbb{R}^2$ (with the normalization condition $\int \varrho(\mathbf{x}, \lambda)d\lambda = 1$). The locally averaged coarse-grained vorticity is expressed in probability terms as

$$\omega^\circ(\mathbf{x}) = \int \varrho(\mathbf{x}, \lambda)\lambda d\lambda, \quad \omega^\circ(\mathbf{x}) = \omega(\mathbf{x}) \quad \text{with } \varrho(\mathbf{x}, \lambda) = \delta(\lambda - \omega(\mathbf{x})).$$

In fact, we introduce a Young measure of levels [27, 28], i.e., a family of probability measures $\lambda = \{\lambda_x\}_{x \in D}$ associated with a sequence of measurable functions $F_k : \mathbb{R}^2 \longrightarrow \mathbb{R}^1$ satisfying the condition

$$\lim_{k \to \infty} \int_D \varrho(\mathbf{x}, F_k(\mathbf{x}))d\mathbf{x} = \int_\Omega \int_{\mathbb{R}^1} \varrho(\mathbf{x}, \tau)(d\lambda_{\mathbf{x}}(\tau))d\mathbf{x}.$$

The function $\omega = \omega^\circ(\mathbf{x})$ is a stationary solution of the two-dimensional Euler–Helmholtz equation (see [29])

$$\frac{\partial\omega}{\partial t} + \mathbf{v}\nabla\omega = 0, \quad \mathbf{v} = \nabla\psi \times \mathbf{e}_z, \quad \psi = -\Delta_2^{-1}\omega, \tag{1}$$

moreover, the stream function must satisfy the condition $\psi|_{\partial D} = 0$ ($\psi$ and $\mathbf{v}$ are consistent with $\omega = \omega^\circ(\mathbf{x})$). Invariants (corresponding to $C_j$ for the microcanonical description) in the evolution of the hydrodynamic system described by (1) are the kinetic energy of the Lamb flow (Hamiltonian function) $E[\omega] = -\frac{1}{2}\int \omega\Delta_2^{-1}\omega d\mathbf{x} = \text{const}$ and a set of Casimir invariants $\mathfrak{C}_{s[\omega]} = \int_D s[\omega(\mathbf{x})]d\mathbf{x}$, where $s[\omega]$ is an arbitrary admissible function (usually, $s[\omega] = \omega^m$, where $m = 1$ and $m = 2$ correspond to physically obvious values of the circulation $\Gamma$ and the enstrophy $W$. For $m \geq 3$, the corresponding moments of the vorticity can be called *generalized enstrophies* of the $m$th order, which characterize the fine local structure of the hydrodynamic system). The conservation of all $\mathfrak{C}_{m \geq 1}$ is equivalent to the conservation of the area of the sublevels $A(\lambda) = \int_D \chi\,(\omega(\mathbf{x}) \leq \lambda)d\mathbf{x}$, where $\chi(D_\lambda)$ is the indicator function of the set $D_\lambda \subset D$.

Following [30, 31], the $\mu$-phase space of the hydrodynamic system is partitioned into cells of two types: "macroscopic" $\{x_1, x_1 + dx_1)_i \times (x_2, x_2 + dx_2)_i$, $i = \overline{1, N}$, and "microscopic"—a single macroscopic cell contains $\ell(\geq 2^2)$ (identical) microcells $D_\ell \times D_\ell$. Let $n_{ij}$ denote the number of microcells occupied by the vorticity level $\lambda_j$ in the $i$th macrocell. Then the number of microstates associated with the macrostate $\{n_{ij}\}$ is

$$w(\{n_{ij}\}) = \prod_j N_j! \prod_i \frac{\ell!}{n_{ij}!},$$

where $N_j = \sum_i n_{ij}$ is the total number of microcells occupied in the level $\lambda_j$ and they satisfy the cell normalization condition $\sum_j n_{ij} = \ell$. This condition specifies the Lynden-Bell statistic of type IV according to the classification in [32] (distinguishable particles with an exclusion principle) and, in vortex statistics, in fact, plays the role of an analogue of the well-known Pauli exclusion principle (taking into account that the $\mu$-phase and configuration spaces of the components of the vortex system coincide in the plane). The entropy of the state of the system with such a structure is computed by taking the logarithm of the function $w(\ell, n)$ and passing to the continual limit in view of the Stirling formula

$$S[\varrho] = -\int \varrho(\mathbf{x}, \lambda)\ln\varrho(\mathbf{x}, \lambda)d\mathbf{x}d\lambda. \tag{2}$$

## 3. EQUILIBRIUM STATES IN STATISTICAL MECHANICS OF THE EULER EQUATION AND APPROACHES TO VARIATION OF FUNCTIONALS IN POISSON SYSTEMS

In the two-dimensional case, under some a priori constraints imposed on the energy and a certain collection of Casimir functionals (see, e.g., [13, 14]), coherent hydrodynamic structures can be associated with solutions of extremum problems for the functionals in the Poisson system corresponding to the Euler equation in the plane.

It seems that the most detailed and clear description of the mathematical apparatus for examining the properties of "quasi-stationary states" of the Euler equation (of medium motions corresponding to a constrained maximum of the system's entropy) can be given within the framework of Miller–Robert–Sommeria (MRS) theory [33, 34]. Here, the basic assumptions are as follows: (i) the evolution of the two-dimensional hydrodynamic flow is described by the Euler equation (with no external forces or dissipation); (ii) the values of all the Casimir invariants are known; and (iii) the system reaches its statistical equilibrium at the most probable state. Under these assumptions, a statistically equilibrium state for the Euler equation in the plane can be obtained by constrained maximization of entropy (2) with additional Lagrange conditions that the Lamb total energy, the total vorticity distribution in the system, and the normalizing condition on the probability density $\varrho(\mathbf{r}, \lambda)$ are preserved.

It should be emphasized that, in the given approach, small-scale structures have no effect on the dynamics of large vortices (since the latter are described using the coarse-grained vorticity and allowance for the fluctuation influence of the small-scale dissipative environment goes beyond the model). Obviously, it is the large vortices that determine the energy of the flow (the Lamb energy of the coarse-grained vorticity is a constant). Thus, the postulates of the MRS model, in fact, coincide with the basic assumptions used in the development of turbulence models [2–5], which are based on the assumption that quasi-deterministic structures exist on large flow scales (see Section 1).

Consider the problem of finding a constrained extremum of the entropy $S[\varrho]$:

$$\mathrm{extr}_{\varrho}\left\{ S[\varrho] \,|\, \int\varrho(\mathbf{x}, \lambda)d\mathbf{x} = A(\lambda), \frac{1}{2}\int\omega^{\circ}\psi d\mathbf{x} = -E, \int\varrho d\lambda = 1\right\},$$

where $\psi = -\Delta_2^{-1}\omega^{\circ}$ is the stream function. In variations, we have

$$\delta S = \beta\delta E + \int\alpha_1(\lambda)\delta A(\lambda)d\lambda + \int\alpha_2(\mathbf{x})\delta\left(\int\varrho(\mathbf{x}, \lambda)d\lambda\right)d\mathbf{x},$$

where $\beta(= 1/T)$ $\alpha_1(\lambda)$, $\alpha_2(\mathbf{x})$ are functional Lagrange multipliers. The corresponding probability density has the form

$$\varrho(\mathbf{x}, \lambda) = (Z(\mathbf{x}))^{-1}\exp(-\alpha_1(\lambda) - \beta\lambda\psi), \quad Z(\mathbf{x}) = \int\exp(-\alpha_1(\lambda) - \beta\lambda\psi)d\lambda.$$

Thus, the coarse-grained vorticity is a function of the variable $\psi$ (therefore, Eq. (1) has the solution $\omega^{\circ}$):

$$\omega^{\circ}(\mathbf{r}) = (Z(\mathbf{r}))^{-1}\int\lambda\exp(-\alpha_1(\lambda) - \beta\lambda\psi(\mathbf{r}))d\lambda \equiv f[\psi(\mathbf{r})].$$

Consider the simplest nontrivial case of a discrete distribution of the coarse-grained vorticity in the system $\omega^{\circ} = P\lambda_1 + (1 - P)\lambda_2$ (where $P$ and $1 - P$ are the probabilities that the vorticity is in a state at the level $\lambda_1$ and $\lambda_2$, respectively). Then the entropy expression becomes $S^{(2)}[\lambda_1, \lambda_2] = -\int(P\ln P - (1 - P)\ln(1 - P))d\mathbf{r}$. For $\lambda_2 \equiv 0$ (vorticity of level $\lambda_1 > 0$ against the background of a vortex-free medium), we have

$$S^{(2)}[\lambda_1, 0] = -\int((\omega^{\circ}/\lambda_1)\ln(\omega^{\circ}/\lambda_1) + (1 - \omega^{\circ}/\lambda_1)\ln(1 - \omega^{\circ}/\lambda_1))d\mathbf{r}.$$

Moreover, the extremum of the entropy is reached on a two-parameter family of equilibrium functions (having a form formally similar to the Fermi distribution in quantum statistics) of the coarse-grained vorticity

$$\omega^{\circ}_{LB}(\mathbf{r}) = \lambda_1(1+\zeta\exp(\beta\lambda_1\psi(\mathbf{r})))^{-1},$$

where $\zeta = \exp(\lambda_1\alpha_1)$. Obviously, in the limit $\omega^{\circ} \ll \lambda_1$ (i.e., as $\ln(1-\omega^{\circ}/\lambda_1) \longrightarrow 0$), we then obtain an analogue of the Boltzmann distribution $\omega^{\circ}_B \equiv \omega^{\circ}_{\beta,\lambda_1}(\mathbf{r}) = C_0\exp(-\beta\lambda_1\psi(\mathbf{r}))$ (and, accordingly, $S^{(2)}[\lambda_1, 0] \longrightarrow S_B = -\int(\omega^{\circ}/\lambda_1)\ln(\omega^{\circ}/\lambda_1)d\mathbf{r}$).

On the basis of what was said above, we can conclude that, in the coarse-grained representation with the effective Lynden-Bell–Pauli principle (which means that the average occupation numbers of a phase microcell are bounded), the Joyce–Montgomery equation for the mean field [35], which describes the stationary dynamics of the vortex system (assuming that it has a unique nontrivial vorticity level), takes the following form in terms of the stream function:

$$-\Delta_2\psi(\mathbf{r}) = \omega^{\circ}_{LB} = \lambda_1\varrho(\mathbf{r},\lambda_1) = \lambda_1(1+\zeta\exp(\beta\lambda_1\psi(\mathbf{r})))^{-1}.$$

In addition to the MRS statistical theory, a rather large number of approaches to the variation of functionals for a Poisson Euler system that take into account various aspects of the problem of existence of statistically equilibrium states can be found in the literature. As major examples, we note the following ones:

(i) the Ellis–Haven–Turkington variational principle [36]

$$\max_{\varrho}\{S_{\chi}[\varrho]\,|\,E, \Gamma, \int\rho d\lambda = 1\};$$

(ii) the Naso–Chavanis–Dubrulle variational principle [37]

$$\max_{\varrho}\{S[\varrho]\,|\,E, \Gamma, \int\rho\lambda^2 d\mathbf{x}d\lambda, \int\rho d\lambda = 1\};$$

(iii) the minimization principle for the coarse-grained enstrophy:

$$\min_{\omega^{\circ}}\left\{\int(\omega^{\circ})^2 d\mathbf{x}\,|\,E, \Gamma|_{\omega^{\circ}(\mathbf{x}) = \int\varrho\lambda d\lambda}\right\};$$

(iv) the maximization principle for the coarse-grained pseudoentropy:

$$\max_{\omega^{\circ}}\{S_g[\omega^{\circ}]\,|\,E, \Gamma\}$$

(here, we used the following notation: $\Gamma = \int\omega^{\circ}d\mathbf{x}$, $S[\varrho] = \int\varrho(\mathbf{x},\lambda)\ln\varrho(\mathbf{x},\lambda)d\mathbf{x}d\lambda$, $S_{\chi}[\varrho] = -\int\varrho(\mathbf{x},\lambda)\ln[\varrho(\mathbf{x},\lambda)/\chi(\lambda)]dd\lambda$, $\chi(\lambda) = \exp\left(-\sum_{k=1}\alpha_k\lambda^k\right)$, and $S_g[\omega^{\circ}] = -\int\mathfrak{C}(\omega^{\circ})d\mathbf{x}$).

In all the variational principles, except for MRS theory, only a distinguished set of quantities associated with the Casimir functionals are assumed to be conserved (in addition to the energy); accordingly, dissipation on small scales, the possibility of a phase transition in the system, etc., can be taken into account.

## 4. CONCLUSIONS

The structure and evolution of large-scale coherent vortices arising in turbulent hydrodynamic flows have to be studied by applying an approach combining numerical experiments and a detailed analysis of the fine structure of these objects (for example, the details of the behavior of the corresponding statistical ensemble). Moreover, it seems necessary to select a collection of physical processes that influence the dynamics of the structures under study (specifically, to separate different scales and to distinguish basic energy-containing structures). Allowance for small-scale turbulence in hydrodynamic computations is a task that can be addressed by applying various approaches (including ones based on kinetic theory), and it has to be treated separately as an important, but not crucial component of the general problem of simulating turbulent flows. In this context, it should be noted that the approach described in this paper can help describe the inertial range of the turbulence spectrum, i.e., in the transition from large to small scales.

It should be noted that, in addition to the above techniques and the determination of coherent structures in hydrodynamic flows by applying the calculus of variations for Poisson systems, we can consider a consistently "microcanonical" approach. For this purpose, it is reasonable to examine the dynamics of elementary vortex (sub)structures from the point of view of Hamiltonian geometry. In this case, the coherence of a composite macrostructure can be thought of as a manifestation of the motion of substructures along geodesics of the Riemannian manifold determined by the metric tensor $g_{ik}$, which is specified as the Hessian of the Hamiltonian function of the vortex system (the collapse and decay of macrostructures are then described in terms of deviations of the corresponding geodesics).

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